Automatic Verification of the Adequacy of Models for Families of Geometric Objects

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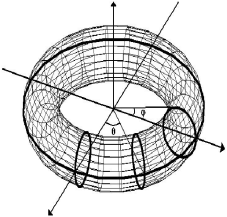
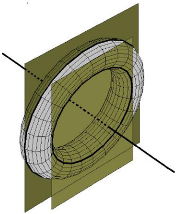
Motivation

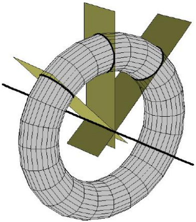
- Surface approximation with **toric splines**: I.e. splines constructed from toric patches



Torus



Circles of Latitude



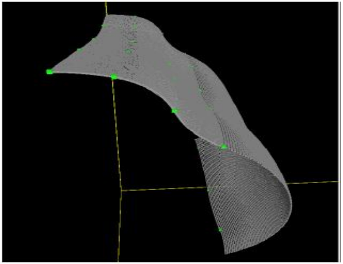
Meridians

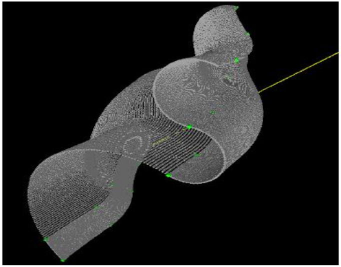
- **Advantages:** Offset invariance of toric patches

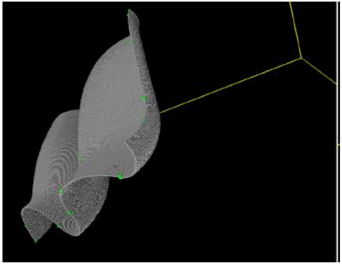
Practical applications:

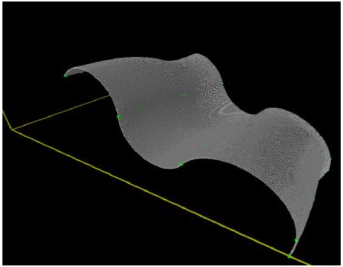
- Modeling of the shrinkage of an aspheric lens (European project FINO)
- Automated estimation of the middle point position of a spheric milling cutter head

Toric splines







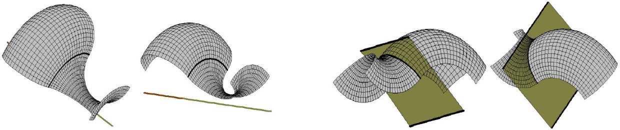


Motivation

Conjecture

Between any two toric patches obtained from distinct tori a $\mathbb{GC}^1$-joint with a regular intersection curve is satisfied only along their lines of curvature

- **Consequence:** If our conjecture is true, the only way of joining toric patches is along their **meridians** and **circles of latitude**



Reduction of conjecture to a statement about points and normals

There is $k \in \mathbb{N}$ such that given a torus T any choice of distinct points $p_1, \ldots, p_k$ with normals $n_1, ..., n_k$, where the points do not lie on a common line of curvature, uniquely describes T

- **Consequence:** We are interested in **automated** methods to prove that certain surfaces are uniquely defined via a finite set of points and corresponding normals

Basic definitions

- A family $G_\chi : \Theta \to P(V)$ of geometric objects over $V = \mathbb{R}^n$ is given by an **index set** $\Theta \subseteq \mathbb{R}^k$ and a **characteristic function**

$$\chi : V \times \Theta \to \{\text{true}, \text{false}\}$$

- The objects in G_χ are

$$G_\chi(\theta) = \{\, v \in V \mid \chi(v, \theta) \,\}$$

- (Θ, χ) is called a **model** of G_χ and each $\theta \in \Theta$ is called a **model parameter**
- A model G_χ is **unambiguous** if for each object $G \neq \emptyset$ in G_χ there is exactly one model parameter $\theta \in \Theta$ such that $G = G_\chi(\theta)$

Example (Spheres in 2-space)

- For spheres we choose $\Theta = V \times \mathbb{R}^+$ and the characteristic function

$$\chi(x, y, c_x, c_y, r) \equiv (x - c_x)^2 + (y - c_y)^2 = r^2$$

- (Θ, χ) is an unambiguous model for the spheres in $\mathbb{R}^2$
- With $\Theta = \mathbb{R}^3$ the model is not unambiguous: For $r \neq 0$

$$G_\chi(x, y, r) = G_\chi(x, y, -r)$$

Outline of our method

Problem in General

- **Given:** An intended description of a family of geometric objects
- **Question:** Does the description specify the objects uniquely?

Our procedure:

- Pick an **unambiguous model** for our object family
- Write down the relationship between the alternative description and the model parameters
- **Automatically** compute a first-order sentence, which is true iff our question can be positively answered
- **Automatically** decide the sentence using real quantifier elimination
- In the positive case **automatically** create a characteristic function of the alternative model
- In the negative case **automatically** generate a counterexample

Example

Example (Spheres in 2-space)

- Unambiguous model for spheres in 2-space: $(V \times \mathbb{R}^+, \chi)$ with

$$\chi(x, y, c_x, c_y, r) \equiv (x - c_x)^2 + (y - c_y)^2 = r^2$$

- ψ says that $p = (p_x, p_y)$ lies on a sphere given by (c_x, c_y, r) and that (n_x, n_y) is a normal vector in p:

$$\begin{aligned} \psi(c_x, c_y, r, p_x, p_y, n_x, n_y) \quad \equiv \quad & (p_x - c_x)^2 + (p_y - c_y)^2 = r^2 \wedge \\ & -(p_x - c_x)n_y + (p_y - c_y)n_x = 0 \end{aligned}$$

- The first-order formula ψ will be called an **intended semantics**
- The vector (n_x, n_y) should be normalized, r must be positive: There should be additional constraints like

$$\tau \equiv r > 0 \quad \text{and} \quad \tau' \equiv n_x^2 + n_y^2 = 1$$

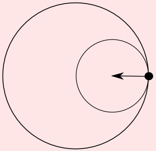
Adequacy

- The reference description (Θ, χ) is an unambiguous model of G_χ
- Intended semantics is called **adequate** if there exists a characteristic function χ^* such that
  - **(i)** G_{χ^*} comprises the same (non-trivial) objects as G_χ
  - **(ii)** Each parameter of Θ' describes a unique object in G_χ

One point with normal is not adequate for spheres

- Intended semantics for spheres by one point and one normal

$$\psi(c_x, c_y, r, p_x, p_y, n_x, n_y) \equiv (p_x - c_x)^2 + (p_y - c_y)^2 = r^2 \wedge -(p_x - c_x)n_y + (p_y - c_y)n_x = 0$$



- Point $(1, 0)$ with normal $(-1, 0)$ describes at least two different spheres
- $\psi(0, 0, 1, 1, 0, -1, 0)$ and $\psi(-1, 0, 2, 1, 0, -1, 0)$

Deciding adequacy

- **Goal: Automatically** create a formula, which decides adequacy given an intended semantics

Theorem (First-order formulation of adequacy)

Let (Θ, χ) be an unambiguous model for G_χ. Let $\Theta' \subseteq \mathbb{R}^l$ with intended semantics ψ with respect to (Θ, χ). Let $\tau(t)$ and $\tau'(t')$ be first-order descriptions of Θ and Θ', respectively. Then the following holds:

(i) The following first-order formula over the reals is equivalent to the adequacy of ψ for G_χ:

$$\Phi(\psi, t, \tau, \tau') \equiv \forall t' \forall t \forall t_0 (\tau \wedge \tau[t_0/t] \wedge \tau' \longrightarrow \psi \wedge \psi[t_0/t] \longrightarrow t = t_0)$$

(ii) In case of adequacy χ^* is given by $\chi^* \equiv \exists t (\tau \wedge \chi \wedge \psi)$ □

- We denote the resulting formula of the above theorem by Φ

Example

Example (Spheres in 2-space)

Question: Is one point with corresponding normal adequate for spheres in 2-space?

- $\tau \equiv r > 0$
- $\tau' \equiv n_x^2 + n_y^2 = 1$
- $\psi(c_x, c_y, r, p_x, p_y, n_x, n_y) \equiv (p_x - c_x)^2 + (p_y - c_y)^2 = r^2$ $\wedge -(p_x - c_x)n_y + (p_y - c_y)n_x = 0$
- **Automatically** generated sentence

$$\begin{aligned}
&\Phi(\psi, \{c_x, c_y, r\}, \tau, \tau') \equiv \\
&\quad \forall p_x \forall p_y \forall n_y \forall n_x \forall c_x \forall c_y \forall r \forall c_{x0} \forall c_{y0} \forall r_0 (r > 0 \wedge r_0 > 0 \wedge n_x^2 + n_y^2 - 1 = 0 \\
&\qquad \longrightarrow (p_x - c_x)^2 + (p_y - c_y)^2 = r^2 \wedge -(p_x - c_x)n_y + (p_y - c_y)n_x = 0 \wedge \\
&\qquad\quad (p_x - c_{x0})^2 + (p_y - c_{y0})^2 = r_0^2 \wedge -(p_x - c_{x0})n_y + (p_y - c_{y0})n_x = 0 \\
&\qquad\quad \longrightarrow c_x - c_{x0} = 0 \wedge c_y - c_{y0} = 0 \wedge r - r_0 = 0)
\end{aligned}$$

Observation:

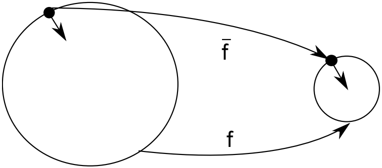
- Our theorem produces **redundancy**: It is sufficient e.g. to consider spheres in the origin as reference ($c_x = c_y = 0$)!

Reducing redundancy

- Choose a group of functions F operating on our object family
- We obtain from F an equivalence relation $\sim_F$ on Θ by defining

$$\theta \sim_F \theta_0 \iff f(G_\chi(\theta)) = G_\chi(\theta_0) \text{ for some } f \in F$$

- We call an intended semantics ψ **compatible** with F if for each $f : V \to V \in F$ there exists a function $\bar{f} : \Theta' \to \Theta'$, which transforms the alternative description the **same way** as f



Lemma (Adequacy in general position)

For an **unambiguous** model and a **compatible** intended semantics the adequacy for a set of representatives of $\Theta/\sim_F$ is equivalent to adequacy for the whole parameter space Θ.

Suitable function groups

- **Suitable function groups** for points and normals are similarities:

$$f : \mathbb{R}^n \to \mathbb{R}^n, \quad f(x) = \alpha A x + b,$$

$\alpha \in \mathbb{R} \setminus \{0\}$, $A \in \mathbb{R}^{n \times n}$ orthogonal, $b \in \mathbb{R}^n$

- The function $\bar{f}$ in definition of compatibility is then

$$\bar{f}(p_1, \ldots, p_k, n_1, \ldots, n_l) = (f(p_1), \ldots, f(p_k), An_1, \ldots, An_l)$$

- **Notice:** Weaker group conditions do not necessarily yield a compatible semantics

Examples (Spheres in 2-space)

- Under the group of similarities we can reduce our reference model to the unit sphere!
- We choose as representative set of $\Theta/\sim_F$ the set $\Theta_F = \{(0, 0, 1)\}$
- **Notice:** We still compare our reference model in general position with arbitrary spheres!

General position

- **Question:** Which coordinates for general position should we choose?
- We choose a set of representatives Θ_0, which has the maximum number of **non changing** coordinates
- For spheres and with respect to similarities that is trivially $\Theta_0 = \{(0,0,1)\}$

Theotem (General position)

Let (Θ, χ) be an unambiguous model of G_χ. Let $\Theta' \subseteq \mathbb{R}^l$ and let ψ be an intended semantics with respect to (Θ, χ). Let $\tau(t)$ and $\tau'(t')$ by the first-order descriptions of Θ and Θ'. Let furthermore ψ be compatible with a group of functions on Θ. Given a set Θ_0 of representants of $\Theta/_\sim$, let σ be a substitution on the variables t in τ defined as follows: **Check in $\Theta_0 \subseteq V = \mathbb{R}^n$ for which coordinates there occurs only one value; σ substitutes that value for the variables in** $\tau(t)$. Then Φ_2 is equivalent to the adequacy of ψ for G_χ:

$$\Phi_2(\psi, t, \tau, \tau', \sigma) \equiv \forall t' \forall t \forall t_0 (\tau\sigma \wedge \tau[t_0/t] \wedge \tau' \longrightarrow \psi\sigma \wedge \psi[t_0/t] \longrightarrow t\sigma = t_0).$$

□

- We denote the resulting formula of the general position theorem by Φ_2

Example

Example (Spheres in 2-space)

Question: Is one point with corresponding normal adequate for spheres in 2-space?

- $\tau \equiv r > 0$
- $\tau' \equiv n_x^2 + n_y^2 = 1$
- $\psi(c_x, c_y, r, p_x, p_y, n_x, n_y) \equiv p_x^2 + p_y^2 = 1 \wedge - p_x n_y + p_y n_x = 0$
- **Automatically** generated sentence

$$\begin{aligned}
&\Phi(\psi, \{c_x, c_y, r\}, \tau, \tau') \equiv \\
&\quad \forall p_x \forall p_y \forall n_y \forall n_x \forall c_{x0} \forall c_{y0} \forall r_0 (r_0 > 0 \wedge n_x^2 + n_y^2 - 1 = 0 \\
&\qquad \longrightarrow p_x^2 + p_y^2 = 1 \wedge -p_x n_y + p_y n_x = 0 \wedge \\
&\qquad\quad (p_x - c_{x0})^2 + (p_y - c_{y0})^2 = r_0^2 \wedge -(p_x - c_{x0}) n_y + (p_y - c_{y0}) n_x = 0 \\
&\qquad\quad \longrightarrow c_{x0} = 0 \wedge c_{y0} = 0 \wedge r_0 = 1)
\end{aligned}$$

- We loose three quantifiers
- We obtain a shorter formula

Deciding adequacy

- **Tool:** Real quantifier elimination by virtual substitution (and CAD in case of degree violations)

$$\exists x \varphi \rightsquigarrow \bigvee_{(\gamma,t)\in E} (\gamma \wedge \varphi[t/x])$$

- **Implementation:** REDLOG (freely available logic system for REDUCE)
- **Generic quantifier elimination:** Assumptions A on non-quantified variables, which simplify elimination

$$A \longrightarrow (\varphi \longleftrightarrow \varphi')$$

- **Extended quantifier elimination:** Sample answers for the outer-most quantifier block (for existential quantifiers) and counter-examples (for universal quantifiers)

$$\exists x(ax \geq 1) \rightsquigarrow \{(a \neq 0, 1/a)\} \quad \forall x(ax \neq 1) \rightsquigarrow \{(a = 0, 1/a)\}$$

- **Worst case complexity** is dominated by quantifier elimination
  - Singly exponential in number of quantifiers, since exclusively universal quantifiers
  - Polynomial in all other parameters (number of atomic formulas, degrees, coefficient sizes)

Examples in 2-space

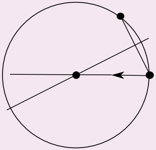
Example (Spheres (continued))

- One point and one normal are not adequate for spheres in 2-space
- REDLOG returns "false" within 20 ms
- Extended quantifier elimination returns in 60 ms the following sample point:

$$\{c_x = -1,\ c_{x0} = 0,\ c_y = 0,\ c_{y0} = 0,\ n_x = 1,\ n_y = 0,\ p_x = -2,\ p_y = 0,\ r = 1,\ r_0 = 2\}$$

- This is the configuration of our counterexample
- We add a second point

$$\begin{aligned}\psi(c_x, c_y, r, p_1, n_1, p_2) \quad \equiv \quad & (p_{1x} - c_x)^2 + (p_{1y} - c_x)^2 = r^2 \wedge \\ & (p_{2x} - c_x)^2 + (p_{2y} - c_x)^2 = r^2 \wedge \\ & -n_{1y}(p_{1x} - c_x) + n_{1x}(p_{1y} - c_y) = 0\end{aligned}$$



- Now the intended semantics is adequate under the non-degeneracy condition $p_1 \neq p_2$

Examples in 2-space

Example (Spheres (continued))

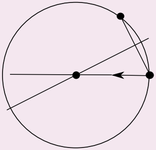
- We drop the quantifiers $\forall p_{1x} \forall p_{1y} \forall p_{2x} \forall p_{2y} \forall n_x \forall n_y$ from Φ_2
- Generic quantifier elimination yields an equivalent formula Φ_2' with 10 atomic formulas subject to the condition

$$A = \{n_x p_{1x} - n_x p_{2x} + n_y p_{1y} - n_y p_{2y} \neq 0,\ n_x \neq 0\}$$

- Regular QE proves $\bigwedge A \longrightarrow \Phi_2'$ in 4.6 s (Φ_2' is redundant, CAD fallback)
- The condition

$$n_x p_{1x} - n_x p_{2x} + n_y p_{1y} - n_y p_{2y} = n_x(p_{1x} - p_{2x}) + n_y(p_{1y} - p_{2y}) \neq 0$$

says that line through p_1 and p_2 is not orthogonal to n



- This follows from $p_1 \neq p_2$

- REDLOG shows in 10 ms that the condition $n_x \neq 0$ is not relevant
- Regular quantifier elimination does not succeed

Examples in 2-space

Example (Circle rings)

- Non-trivial intersection of a torus with a plane, which has the rotation axis of the torus as its normal
- Consists of two circles with a common center but distinct radii
- Characteristic function:

$$\begin{aligned}\chi(x, y, c_x, c_y, r_1, r_2) \quad \equiv \quad & (x - c_x)^2 + (y - c_y)^2 = (r_1 - r_2)^2 \vee \\ & (x - c_x)^2 + (y - c_y)^2 = (r_1 + r_2)^2\end{aligned}$$

- **Modified implementation:** Generic elimination admits only assumptions where the total degree of the left hand side polynomials is at most 2
- In addition, we do not admit any assumption on n_{1x}
- Way, we obtain after 15.6 s a quantifier-free equivalent Φ_2' with 230 atomic formulas subject to the non-degeneracy conditions:

$$\mathrm{A} = \{n_{1y} \neq 0,\ n_{2x}p_{1y} - n_{2y}p_{1x} \neq 0,\ n_{2x} \neq 0,\ p_{1y} \neq 0\}$$

- Automated proof of $\bigwedge \mathrm{A} \longrightarrow \Phi_2'$ fails

Examples in 3-space

Example (Planes)

- Planes are uniquely defined by one point a corresponding normal
- Model parameter restrictions $\Theta = \{\, \theta \in \mathbb{R}^4 \mid \tau(\theta) \,\}$, where

$$\tau(a,b,c,d) \equiv \|(a,b,c)\|^2 = 1 \wedge (a > 0 \vee (a = 0 \wedge b > 0) \vee (a = 0 \wedge b = 0 \wedge c > 0))$$

- Intended semantics:

$$\psi(p,n,a,b,c,d) \quad \equiv \quad \langle p \mid (a,b,c)\rangle + d = 0 \wedge n \times (a,b,c) = 0,$$

- With respect to the similarities

$$\Theta_0 = \{(0,1,0,0)\} \quad \text{with} \quad \sigma = [0/a, 1/b, 0/c, 0/d]$$

- Regular quantifier elimination says “true” in less than 10 ms
- By means of generic quantifier elimination we obtain within less than 10 ms a quantifier-free

$$\chi' \equiv \langle n \mid p - (x,y,z)\rangle = 0$$

subject to the condition $\mathrm{A} = \{n_x \neq 0\}$ (monomial assumptions)

Examples in 3-space

Example (Spheres)

- An unambiguous model for spheres is $\Theta = \mathbb{R}^3 \times \mathbb{R}^+$, where $(c_x, c_y, c_z, r) \in \Theta$
- Two points and two normals: $p_1 = (p_{1x}, p_{1y}, p_{1z})$, $p_2 = (p_{2x}, p_{2y}, p_{2z})$, and $n = (n_x, n_y, n_z)$
- Intended semantics

$$\psi(c, r, p_1, p_2, n) \equiv \|p_1 - c\|^2 = r^2 \wedge \|p_2 - c\|^2 = r^2 \wedge (p_1 - c) \times n = 0,$$

- Parameter space restriction of Θ': $\tau'(p_1, p_2, n) \equiv \|n\|^2 = 1 \wedge p_1 \neq p_2$
- We drop the outermost block: $\forall p_{1x} \forall p_{1y} \forall p_{1z} \forall p_{2x} \forall p_{2y} \forall p_{2z} \forall n_x \forall n_y \forall n_z$
- Generic quantifier elimination yields within 30 ms the quantifier-free equivalent "true" subject to the conditions

$$A = \{n_x \neq 0, n_y \neq 0, p_{1x} \neq p_{2x}\}$$

- Regular quantifier elimination on Φ does not succeed within reasonable time and space
- Generic quantifier elimination on Φ_2 generated according to the general position theorem delivers a more complicated result

Examples in 3-space

Example (Tori)

- Parameter space:

$$\begin{aligned}\tau(c, r, r_1, r_2) \quad \equiv \quad & r_1 > r_2 \wedge r_2 > 0 \wedge \|r\| = 1 \wedge \\ & (r_x > 0 \vee (r_x = 0 \wedge r_y > 0) \vee (r_x = 0 \wedge r_y = 0 \wedge r_z = 1))\end{aligned}$$

- Characteristic function:

$$\begin{aligned}\chi(c, r, r_1, r_2, p_i) \quad \equiv \quad & \exists r'_x \exists r'_y \exists r'_z (\|r'\|^2 = 1 \wedge \langle r \mid p_i + r_2 r' - c\rangle = 0 \\ & \wedge \langle r \times r' \mid c - p_i\rangle = 0 \wedge \|p_i + r_2 r' - c\|^2 = r_1^2).\end{aligned}$$

- One point with one normal on a torus

$$\begin{aligned}\omega_i(c, r, r_1, r_2, p_i, n_i) \quad \equiv \quad & \exists r'_x \exists r'_y \exists r'_z (\|r'\|^2 = 1 \\ & \wedge \langle r \mid p_i + r_2 r' - c\rangle = 0 \wedge \langle r \times r' \mid c - p_i\rangle = 0 \\ & \wedge \|p_i + r_2 r' - c\|^2 = r_1^2 \wedge \exists \lambda (n_i = \lambda r'))\end{aligned}$$

- Intended semantics for k points and k corresponding normals $\varphi \wedge \bigwedge_{i=1}^{k} \omega_i$
- Conditions φ should imply that the points **do not** lie on a line of curvature
- Quantifier elimination does not succeed for each k

Conclusions

- For families of geometric objects we automatically generate a first-order formula that can be used to automatically decide via real quantifier elimination whether a given alternative description of the object family is suitable to represent these geometric objects uniquely
- In the **positive case**: We automatically generate by real quantifier elimination a quantifier-free description of the new characteristic function
- In the **negative case**: We automatically obtain by extended quantifier elimination a counterexample for the uniqueness
- Simplification of the resulting first-order formulas by transformation of the reference objects into “general position”
- Worst-case complexity of our approach
- Applications of our framework to several non-trivial examples in real 2-space and real 3-space
- New challenging benchmark example: Find necessary and sufficient conditions on finitely many points on a torus and normals in these points such that they uniquely describe a torus